Triangular Polynomial Notation

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Abstract

In this article we will discuss a new type of notation for homogenous polynomials of 3 variables, and its applications in solving Olympiad inequalities using the AM-GM inequality, Muirhead's Inequality, and Schur's Inequality. I suggest reading [1] first for a clearer explantion on the mechanics of this notation. This article is more focused on the applications to Olympiad inequalities

1 Triangular Notation

Consider a triangle made up of n rows, where there are k entries on the kth row. This is similar to how Pascal's triangle is written. Label one corner (not the entry!) as a, the other 2 as b and c. (Figure 1) The degree of the polynomial is the side length (number of entries on each side) of the triangle.

```
          a
          0
       0     0
    0     0     0
 0     0     0     0
b                    c
```

In the Triangular Notation, we fill in the coefficients of the terms as entries. Label each entry in the triangle with a ordered triple, which states the number of rows (can be taken in any of the three directions) that are below it plus one with respect to a certain corner; a, b, and c in that order. Refer to the ordered triples place on the entries of the triangle below. We let the entry with ordered triple (x_a, x_b, x_c) be filled with the coefficient of the term in the polynomial with variable term $x^{x_a}x^{x_b}x^{x_c}$. An entry may be filled with the number 0. (Figure 2)

```
                      a
                   (3,0,0)
            (2,1,0)         (2,0,1)
     (1,2,0)        (1,1,1)         (1,0,2)
(0,3,0)     (0,2,1)         (0,1,2)         (0,0,3)
b                                                   c
```

Note that this notation is similar to the Barycentric Coordinate system, with the reference triangle be the triangle made by the three corners.

In Figure 3, the polynomial represented is $x^2y + y^2z + z^2x$, while in Figure 4, the polynomial is $2x^2 + 3y^2 + z^2 + xy + 2yz$.

```
         0
      1     0
   0     0     1
0     1     0     0
```

Figure 3

```
      2
   1     0
3     2     1
```

Figure 4

When solving a homogenous polynomial inequality, expand and express both sides in this notation and then subtract one from the other. Most of the time, we want to show that the difference is greater than or equal to 0.

2 Addition, Subtraction, Multiplication (no Division)

2.1 Addition

Add each term to each corresponding term.
Example:

```
      1                     3                    4
   2     2      +        2     2         =    4     4
1     2     1         1     5     4        2     7     5
```

```
        12                     63                       75
    21      23        +    24      76         =     45      99
56      54      34      53     12      86       109     66      120
```

2.2 Subtraction

Subtract each term to each corresponding term Example:

```
         1                         23                              -22
      2     7                  43      3                       -41      4
   3     2     8       -    65     12     -42       =      -62     -10      50
4     5     9     10     78    64    -12     -1        -74     -59     21
```

2.3 Multiplication

See [1] for an explanation of it.

3 Solving Inequalities with the Triangular Notation

If we are trying to prove that a quatity X is greater than a quantity Y, this is analogous to showing that $X - Y \geq 0$. Hence, to only deal with one term, we only consider the quantity $X - Y$ and show wthat it is ≥ 0. The rest of this section shows how to deal with the single quantity that you want to show that it is greater than or equal to 0. The best way to do so is by examples.

We first consider the two basic inequalities: Muirhead's Inequality and Schur's Inequality. Note that AM-GM is simply a special case of Muirhead's, so let's place that under Muirhead's. Look at [2] for the statements of the inequalities and the symmetric sum notation.

Exercise: Visualize both of the inequalities in the Triangular Notation. The Schur's is especially pretty!

But note that symmetric sums can be combined with AM-GN to be even more powerful! For example, if we consider each term separately. For example, $[5, 3, 0] + [3, 1, 0] \geq 2[4, 2, 0]$.

3.1 Spotting Inequalities

After coming up with the quantity that we wish to be nonnegative, construct inequalities such that when added up, gives the result that the quantity is nonnegative.

The recommended way is to find an inequality that has same equality case as the one that we are working with, and then subtract it from the current inequality.

Repeatedly do it until we get a triangle with all zeros. Afterwards, collect all the inequalities used.

3.2 Writing a Solution

After knowing which inequalities are used, collate them. The best way to write the solution is to write it in standard notation, writing the terms from left to write. Writing the solution using this method is not recommended because it is not well-known, and if you will use it in your solution, you still have to explain this notation, which takes time. Once you have the inequalities, you just have to state them and prove them.

Write down the clearing of denominators, resulting terms, and the final quantity that we wish to be nonnegative. It is permissible to write those using the regular expanded notation or the cyclic/symmetric sum notation. However, if you will be using the cyclic/symmetric sum notation, be careful as they might be confused. A cyclic sum over k variables has k terms, while a symmetric sum over k variables has $k!$ terms. In a symmetric sum, some of the terms may be repeated.

Afterwards, state the inequalities that you wish to use, together with their concise proofs (i.e. this inequality is true by Muirhead's as [4,0,0] majorizes [3,1,0]). Do not forget to write the majorization condition. The inequalities have to be written one by one, and at the end, sum up all of them, then state that it implies that our desired quantity is greater than or equal to zero.

4 Tips and Strategies

4.1 Strategies

This notation is very helpful... but how do we even get to use it? In this subsection, I will discuss ways to convert inequalities to those that could be written with this method.

4.1.1 Ravi's Substitution

Ravi's Substitution is used when there is a condition that the variables a, b, and c are the sides of a triangle. In the substitution, we substitute $a = y + z$, $b = x + z$, and $c = x + y$. This is necessary to be able to use the notation because once the quantity is expanded using the notation, it is usually hard to revert back into a factored form. Also, using the substitution most of the time gives an inequality that could be easily proven using Muirhead's / Schur's. See [4] for a proof and an in-depth discussion of Ravi's Substitution

4.1.2 Making both sides polynomials

To apply the notation, both of the sides have to be polynomials. Thus, we will have a problem if there is a denominator in either of the sides. To do this, as long as there are no square roots, we multiply both sides by the least common multiple of the denominators. The result is are polynomials for both sides.
Example: $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$ is equivalnt to $a^2b + b^2c + c^2a \geq 3abc$.

4.1.3 Limiting inequalities to Muirhead's, Schur's, SOS, and symmetric inequalities

Those are the inequalities usually useful for dealing with terms of the same degree. Note that while Cauchy-Schwarz is very useful in inequalities at the start, it relies on a nice splitting of terms. However, this no longer exists, usually, once the quantity is expanded to the notation. Clearly, inequalities on convex also do not work well as it is hard to define functions after the quantity has been expanded.

Clearly, having square roots would be a big problem. When that happens, it is best to clear the sqaure roots using the QM-AM ineqality, the Cauchy-Schwarz Inequality, or others. Remember that the square roots have to be cleared first before this notation can be used.

However, it is possible to set the variables as $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{c}$, or even $a^{\frac{1}{3}}$, $b^{\frac{1}{3}}$, $c^{\frac{1}{3}}$.

4.1.4 The Outer Terms

We are trying to prove that a certain quantity is nonnegative. According to Muirhead's Inequality, the more outer terms in the expanded quantity, and when you take the sum of them across 3 variables, those are the ones which are greater than the inner terms. Hence they are more dangerous when they are negative. If there are terms more "outer" than those, then deal with those immediately before dealing wiht the inner terms. Note that it is not possible to use "inner" terms to cancel out "outer" terms.

For example, if we want to prove that this is nonnegative, we have to consider the outer -1s first before the inner terms. Once we are done with this, we could not remove the inner terms.

A good candidate to remove negative outer terms is Schur's Inequality, as 6 outer terms are removed (at the expense of only 3 more-outer terms). In addition, 3 inner terms, which are smaller, are also removed. Muirhead's usually cannot stand alone when there are lots of negative outer terms. this is where Schur's inequality comes in. However, one of the pitfalls of this notation is when both Muirhead's and Schur's (a.k.a. Schurhead) do not work. In this case, it is advisable to defer to SOS. [3]. A nice example could be seen below.

4.1.5 Normalizing

To use the notation, it is necessary for all the terms to be of the same degree. Now there will be a problem if the terms are not of the same degree. However, more of the time that this happens, the inequality usually comes with a nonhomogenous condition. We could move terms from side to side such that we can devise of a quick way to add or subtract degrees from terms. That is only when we can evaluate the sum and then express it in the notation.

The simplest kind to normalize are those with condition Prove that ... for all ... such that $x + y + z = 1$. With this, we could substitute terms of degree 0 with degree 1 or vice versa. This is relatively simple.

Another simpler kind to normalize is the ones with the product condition, i.e. $xyz = 1$. A helpful substitution is $x = \frac{a}{b}$, $y = \frac{b}{c}$, and $z = \frac{c}{a}$. Clearly $xyz = \frac{abc}{bca} = 1$. Another helpful, but less popular substitution is $x = \frac{bc}{a^2}$, etc. The former is used to keep the degree low, while the latter is used to maintain symmetry.

Example: Normalize the inequality $(x^2+y^2+z^2)-(x+y+z) \geq 0$, if we have $x+y+z = 3$. to do this, we multiply $x + y + z$ by $\frac{x+y+z}{3}$. Then all of the terms in the LHS will be of degree 2 and the condition is also satisfied.

4.1.6 Defining new variables: forcing the equality case

While the notation can technically still be used even though the equality case is not $a = b = c$, it will not be as useful. The main use of the notation is to see the symmetric / cyclic terms and how do they cancel out. If the equality case is not $a = b = c$, then using Muirhead's / Schur's directly would be misleading as they require $a = b = c$. Thus, we could define new variables, a', b', c', such that $a' = b' = c'$.

For example, if we want to show that $9a^2 + 4b^2 + c^2 - 12ab - 4bc - 6ac \geq 0$, putting this directly to our notation would not give much. It is better to perform the quite-intuitive substitution $3a = a'$, $2b = b'$, $c = c'$, to make it $a'^2 + b'^2 + c'^2 - a'b' - b'c' - c'a' \geq 0$. In our notation, it is

$$\begin{array}{ccccc} & & 1 & & \\ & -1 & & -1 & \\ 1 & & -1 & & 1 \end{array}$$

and the application of Muirhead's can be more clearly seen. It is equivalent, when doubled, to $[2, 0, 0] \geq [1, 1, 0]$.

4.1.7 Using knowledge about cyclicity / symmetry

If you know that an expression is symmetric wrt the variables a and b, it is possible to just fill in the symmetric terms' coefficients without further computations. This will save lots of time.

4.2 Tips

This notation can be messy if not done properly. Here are some tips to make solving inequalities with this notation much easier.

4.2.1 Building your intuition

After expanding the whole polyomial inequality in the notation, it may be confusing where to begin. Even though the entries are presented in a much clearer fashion, it may still be difficult to find the needed inequalities. The best way to do this is practice using the notation. When practicing, find inequalities without square roots. Afterwards, expand it and try to find the necessary applications of the inequalities. See [5] for lots of problems

4.2.2 Knowing its limitations

Clearly this notation cannot do everything, especially those with square roots. Once you see an inequality with square roots and you badly want to use this notation, check first if it can be made into a polynomial. If it can, then do such, normalize if needed, then use the notation. If there seems to be no viable way, then do not force it. The time may be better spent finding the "proper approach". (Note: this approach is not "improper", but it is not usually the expected solution. Most inequalities have beautiful solutions, and expanding everything is almost always not one of them.

Unfortunately, it is not only the square roots that could cause problems. Weird conditions like $abc = a + b + c + 2$ seem to have no way out of it, as terms of degree 3, 1, and 0 are involved. If you want to know how to work through conditions and substitutions, read chapter 1 of [6]. However it is still best to use your common sense on which can be done with this and cannot.

There are some inequalities, on the other hand, which can be expressed in this notation, but will be hoorendously impractical. If the degree is going above 10, and the terms are going to the ten-thousands, hundred-thousands, or even millions, it may be best to give up with this notation. In those cases, it might be hard to find the necessary inequalities. Even after solving it, writing the solution will be another big problem.

Basically, know the limitations of this notation. It can be very helpful, but going too far with it is not helpful at all.

4.2.3 Checking Calculations

Most of the time, the process of expanding has many steps. In the more complicated problems, the coefficient of terms can go up to the hundreds or even thousands. The best way is to ensure the sum of coefficients "makes sense" in every step. After finishing every step, calculate the sum of coefficients. See if it matches the sum of coefficients when written in the notation. They have to match. Otherwise, something went wrong.

4.3 Scratchwork Organization

If you are using this notation with paper and pen, the scratchwork has to be organized. It is very likely that the degree of the polynomial could go up to 10 and the terms in the thousands, so it is advisable to use 1 paper per triangle diagram. Likewise, leave a big space in between rows so that sums can be clearly seen, especially in multiplication!

4.3.1 Multiplying Terms

When multiplying terms, give a large space in the scratch paper for it. Instead of writing each of the distributed sums one by one, just do it in one large triangle. Take it one sum at a time, so that you would not get lost. If there is an error here, however, you have to redo the whole multiplication as it is nearly impossible to spot the error as there are many entries written in the same blank. See the Solutions for some examples.

4.3.2 Cyclic Adding Shortcuts

It is possible to divide the triangle into three parts which are cyclically identical. Once you are done computing for one of the parts, you could cyclically fill up the rest

4.3.3 Checking Coefficients

After every step, calculate the supposed sum of coefficients and the sum of coefficients in the resulting expression. They must be equal. If they are unequal, there is something wrong.

5 Examples

In this section I will work through 2 inequalities. They have already been reduced to the form $X \geq 0$. We are going to show that the given expression is nonnegative

5.1 Example 1

			2			
		-1		-1		
	-1		0		-1	
2		-1		-1		2

There are 3 ways to approach this. One is to notice that the expression is basically $[3,0,0] - [2,1,0] \geq 0$. But that is equivalent to $[3,0,0] \geq [2,1,0]$. This is true by Muirhead's as $[3,0,0]$ majorizes $[2,1,0]$.

The second way is to break up the sum by the 3 outer rows. Each of them is a 2-variable application of Muirhead's on $a^3 + b^3 \geq a^2b + ab^2$ and similar cyclic terms. This, again, is true by Muirhead's as $[3,0]$ majorizes $[2,1]$.

The third way is much more complicated. It is possible to use Schur's Inequality, which is

			1			
		-1		-1		
	-1		3		-1	
1		-1		-1		1

We then subtract it from the original quantity, to get

			1			
		0		0		
	0		-3		0	
1		0		0		1

and then it becomes the AM-GM inequality on 3 variables: $a^3 + b^3 + c^3 \geq 3abc$.

5.2 Example 2

$$
\begin{array}{ccccccccccccc}
 & & & & & & 8 & & & & & & \\
 & & & & & 4 & & 4 & & & & & \\
 & & & & 1 & & 10 & & 1 & & & & \\
 & & & 10 & & -26 & & -26 & & 10 & & & \\
 & & 1 & & -26 & & 42 & & -26 & & 1 & & \\
 & 4 & & 10 & & -26 & & -26 & & 10 & & 4 & \\
8 & & 4 & & 1 & & 10 & & 1 & & 4 & & 8
\end{array}
$$

The main difficulty here is the positive 42 at the center. A way to deal with this is Schur's Inequality. Look at the 2 applications of Schur's Inequality below. Both of the values are at least zero. 42 is too big to be finished by one application of Schur's, as for one application only, the terms have to be at least 14, but there are no positive terms that are at least 14.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 0 & & & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & 0 & & 0 & & 0 & & & & \\
 & & & 10 & & -10 & & -10 & & 10 & & & \\
 & & 0 & & -10 & & 30 & & -10 & & 0 & & \\
 & 0 & & 0 & & -10 & & -10 & & 0 & & 0 & \\
0 & & 0 & & 0 & & 10 & & 0 & & 0 & & 0
\end{array}
$$

$$
\begin{array}{ccccccccccccc}
 & & & & & & 0 & & & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & 0 & & 10 & & 0 & & & & \\
 & & & 0 & & -10 & & -10 & & 0 & & & \\
 & & 0 & & -10 & & 30 & & -10 & & 0 & & \\
 & 0 & & 10 & & -10 & & -10 & & 10 & & 0 & \\
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}
$$

Subtracting their sum from the original triangle gives

$$
\begin{array}{ccccccccccccc}
 & & & & & & 8 & & & & & & \\
 & & & & & 4 & & 4 & & & & & \\
 & & & & 1 & & 0 & & 1 & & & & \\
 & & & 0 & & -6 & & -6 & & 0 & & & \\
 & & 1 & & -6 & & -18 & & -6 & & 1 & & \\
 & 4 & & 0 & & -6 & & -6 & & 0 & & 4 & \\
8 & & 4 & & 1 & & 0 & & 1 & & 4 & & 8
\end{array}
$$

But then notice that it is equal to $4\sum_{sym} a^6 + 4\sum_{sym} a^5 b + \sum_{sym} a^4 b^2 - 6\sum_{sym} a^3 b^2 c - 3\sum_{sym} a^2 b^2 c^2$, or $4[6,0,0] + 4[5,0,0] + [4,2,0] \geq 6[3,2,1] + 3[2,2,2]$.

We know by Muirhead's that $4[6,0,0] \geq 4[3,2,1]$, $2[5,1,0] \geq 2[3,2,1]$, $2[5,1,0] \geq 2[2,2,2]$, $[4,2,0] \geq [2,2,2]$. Adding all of these up gives that $4[6,0,0] + 4[5,0,0] + [4,2,0] \geq 6[3,2,1] + 3[2,2,2]$ is nonnegative.

6 Problems

These are problems from various sources. They are arranged roughly in increasing order of difficulty.

1. (Japan TST 2004): If $a+b+c=1$, $\frac{1+a}{1-a}+\frac{1+b}{1-b}+\frac{1+c}{1-c}\leq\frac{2a}{b}+\frac{2b}{c}+\frac{2c}{a}$.
2. If $a+b+c=1$, $(\frac{1}{a}+1)(\frac{1}{b}+1)(\frac{1}{c}+1)\geq 64$.
3. If $a+b+c=1$, $(\frac{1}{a}-a)(\frac{1}{b}-1)(\frac{1}{c}-1)\geq 8$.
4. (Romania ???): If $a+b+c=3$, $a^2+b^2+c^2+abc\geq 4$.
5. (Mildorf): If $a+b+c=1$, $a^3+b^3+c^3+6abc\geq\frac{1}{4}$.
6. If $a+b+c=1$, $5(a^2+b^2+c^2)\leq 6(a^3+b^3+c^3+1)$.
7. (APMO 1998): $\left(1+\frac{x}{y}\right)(\left(1+\frac{y}{z}\right)\left(1+\frac{z}{x}\right)\geq 2+\frac{2(x+y+z}{\sqrt[3]{xyz}}$.
8. (Secrets in Inequalities): $\frac{a^3}{a^3+b^3+abc}+\frac{b^3}{b^3+c^3+abc}+\frac{c^3}{c^3+a^3+abc}$.
9. (Secrets in Inequalities): If $abc=1$, $\frac{1}{a^2+a+1}+\frac{1}{b^2+b+1}+\frac{1}{c^2+c+a}\geq 1$.
10. (Mildorf): If $abc=1$, $a+b+c\leq a^2+b^2+c^2$.
11. (Belarus 1999): If $a^2+b^2+c^2=3$, $\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ca}\geq\frac{3}{2}$.
12. (Bulgaria TST 2003): If $a+b+c=3$, show that $\frac{a}{1+b^2}+\frac{b}{1+c^2}+\frac{c}{1+a^2}\geq\frac{3}{2}$.
13. $\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\geq\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+9$.
14. (Japan 1997): $\frac{(b+c-a)^2}{(b+c)^2+a^2}+\frac{(c+a-b)^2}{(c+a)^2+b^2)}+\frac{(a+b-c)^2}{(a+b)^2+c^2)}\geq\frac{3}{5}$.
15. (Mathematical Reflections): $\frac{ab}{3a+4b+2c}+\frac{bc}{3b+4c+2a}+\frac{ca}{3c+4a+2b}\leq\frac{a+b+c}{9}$.
16. (Problems from the Book): If $xy+yz+zx+2xyz=1$, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq 4(x+y+z)$.
(Hint: you need a substitution to apply the notation)

7 References

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