# Smallest Area of a Triangle Formed from the Tangent Line of a Parabola

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# Introduction

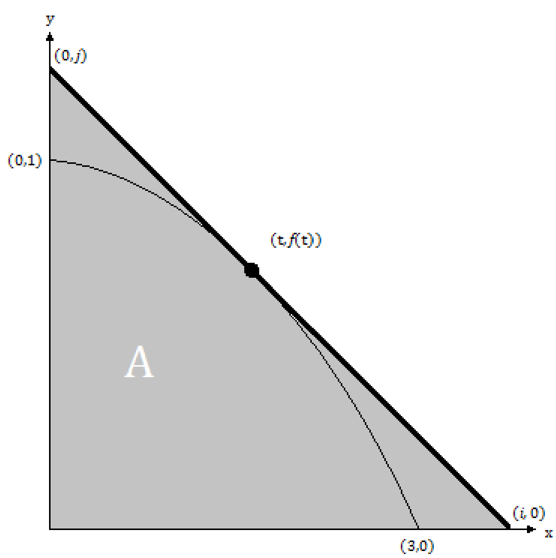
This problem is an applied optimization problem. The problem is to minimize the area of the triangle formed by a tangent line to the function $y = 1 - \frac{1}{9}x^2$. The triangle is defined by the origin, the x-intercept of the tangent line, and the y-intercept of the tangent line. Only triangles formed in the first quadrant are of concern.

# Variables

The original parabola is represented by the function $f$. The relevant variables are the area, represented by $A$; the x-intercept, represented by $i$, and the y-intercept, represented by $j$. The x-coordinate of the tangency point is represented by $t$, so the y-coordinate of the tangency point is represented by $f(t) = 1 - \frac{1}{9}t^2$. The slope of the tangent line, then, is represented by $f'(t) = -\frac{2}{9}t$. The x-intercept of $f$ is evaluated below.

$$0 = 1 - \frac{1}{9}x^2$$
$$x^2 = 9$$
$$x = \pm 3$$
$$x = 3 \text{ as we are focused on the first quadrant.}$$

A diagram is drawn below.

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## Objective Function and Range

First, we must find the objective function $A(t)$ and range constraining $t$.

### Objective Function

The objective function is $A(t)$, or the area of the triangle as dependent on $t$, the arbitrary x-coordinate on the function $f$. $A(t) = \frac{1}{2} \cdot ij$ by the formula for the area of a triangle. The tangent line in slope-intercept form is

$y - f(t) = f'(t)(x - t)$ by the definition of a tangent line.
$y - (1 - \frac{1}{9}t^2) = -\frac{2}{9}t(x - t)$ by substitution.
$$y - 1 + \frac{1}{9}t^2 = -\frac{2}{9}tx + \frac{2}{9}t^2$$
$$y = -\frac{2}{9}tx + \frac{1}{9}t^2 + 1$$
Now we must evaluate for the intercepts $i$ and $j$.
$$0 = -\frac{2}{9}ti + \frac{1}{9}t^2 + 1$$
$$\frac{2}{9}ti = \frac{1}{9}t^2 + 1$$
$$i = \frac{9}{2t} \cdot \frac{t^2}{9} + \frac{9}{2t}$$
$$i = \frac{t}{2} + \frac{9}{2t}$$

$$j = -\frac{2}{9}t \cdot 0 + \frac{1}{9}t^2 + 1$$
$$j = \frac{1}{9}t^2 + 1$$
Now that we have $i$ and $j$ in terms of $t$, we can use them to find the area function or objective function $A(t)$.
$A(t) = \frac{1}{2} \cdot ij$ by the definition of the area of a triangle.
$A(t) = \frac{1}{2} \cdot (\frac{t}{2} + \frac{9}{2t}) \cdot (\frac{1}{9}t^2 + 1)$ by substitution.
$A(t) = \frac{t^3}{36} + \frac{t}{2} + \frac{9}{4t}$ by expansion.

### Range

The range of $t$ is constrained to the first quadrant. It ranges from 0 to the x-intercept of $f$, which, as evaluated above, is 3. Thus, $t \in [0, 3]$.

## Optimization

To optimize $A(t)$, we must first find $A'(t)$.

$A'(t) = \frac{t^2}{12} + \frac{1}{2} - \frac{9}{4t^2}$ by the Power Rule.
Then, we must set the function to 0 to find points of interest.
$$\frac{t^2}{12} + \frac{1}{2} - \frac{9}{4t^2} = 0$$
$t^4 + 6t^2 - 27 = 0$ by multiplying by $12t^2$.
$$(t^2 + 9)(t^2 - 3) = 0$$
$t = \pm\sqrt{3}$ if we solve for $t$.
$t = \sqrt{3}$ since we are working in the first quadrant.

Now we must analyze this critical point and the endpoints. The points are 0, $\sqrt{3}$, and 3.

$$A(0) = \frac{9}{4 \cdot 0}$$
$$A(0) = \infty$$

$$A(\sqrt{3}) = \frac{\sqrt{3}^3}{36} + \frac{\sqrt{3}}{2} + \frac{9}{4\sqrt{3}}$$
$$A(\sqrt{3}) = \frac{\sqrt{3}}{12} + \frac{6\sqrt{3}}{12} + \frac{9\sqrt{3}}{12}$$
$$A(\sqrt{3}) = \frac{16\sqrt{3}}{12}$$
$$A(\sqrt{3}) = \frac{4\sqrt{3}}{3}$$

$$A(3) = \frac{3^3}{36} + \frac{3}{2} + \frac{9}{4 \cdot 3}$$
$$A(3) = \frac{3}{4} + \frac{6}{4} + \frac{3}{4}$$
$$A(3) = \frac{12}{4}$$
$$A(3) = 3$$

As seen, $t = \sqrt{3}$ yields the smallest area of $\frac{4\sqrt{3}}{3}$. Now we must calculate the intercepts $i$ and $j$, the y-coordinate of the tangent point $f(t)$, and the equation of the tangent line in point slope form $y - f(t) = f'(t)t(x - t)$.

$$i = \frac{t}{2} + \frac{9}{2t}$$
$$i = \frac{\sqrt{3}}{2} + \frac{9}{2\sqrt{3}}$$
$$i = \frac{\sqrt{3}}{2} + \frac{3\sqrt{3}}{2}$$
$$i = \frac{4\sqrt{3}}{2}$$
$$i = 2\sqrt{3}$$

$$j = \frac{1}{9}t^2 + 1$$
$$j = \frac{1}{3} + 1$$
$$j = \frac{4}{3}$$

$$f(\sqrt{3}) = 1 - \frac{1}{3}$$
$$f(\sqrt{3}) = \frac{2}{3}$$

$$y - \frac{2}{3} = -\frac{2\sqrt{3}}{9}(x - \sqrt{3})$$

# Conclusion

In conclusion, the triangle with the smallest area is the one with the following properties. The tangent point is $(\sqrt{3}, \frac{2}{3})$. The equation of the tangent line is $y - \frac{2}{3} = -\frac{2\sqrt{3}}{9}(x - \sqrt{3})$. The line's intercepts are $(2\sqrt{3}, 0)$ and $(0, \frac{4}{3})$. The area of the triangle is $\frac{4\sqrt{3}}{3}$.