In this Investigative Project, you will use l'Hôpital's Rule to develop a hierarchy of functions. Hopefully you've carefully thought about the questions and ideas posed at the end of the in-class worksheet on l'Hôpital's Rule. In the beginning, you compared the functions $f(x) = x^2$ and $g(x) = 2^x$. Include a careful write-up of this analysis here.

You should have discovered that $\lim_{x\to\infty} \frac{2^x}{x^2} \to \infty$. This means that the function 2^x is much larger "in the long run" than the function x^2. We can convey this in symbols as

$$x^2 \underset{\to}{\leq} 2^x.$$

In this case, the exponential function "wins" this battle, but is that always the case? What happens if we have a larger power for x in the power function and/or a smaller base in the exponential function? Is $x^{100} \underset{\to}{\leq} 1.5^x$? Try comparing a few different pairs of one power function and one exponential function. How low can you make the base of the exponential and/or how high can you make the power while maintaining the same $\underset{\to}{\leq}$ ordering? Determine a theorem, given it a name, and provide (if not a proof) a strong argument for your theorem. It might have a form that looks like:

> **The __________________ Theorem:** For any value p where ___ $< p <$ ___ and/or any value of b where ___ $< b <$ ___,
>
> $$\lim_{x\to\infty} \frac{b^x}{x^p} \to \infty,$$
>
> so $x^p \underset{\to}{\leq} b^x$.

and you will reference l'Hôpital's Rule in the justification. Note that your conditions on p and b maybe in a different form (like $b > 4$) or there may be no restrictions on one or the other (for any real numbers p and $b \ldots$). It is your job to try to push these bounds as far as you can.

The activity outlined above compared the end behavior of exponential functions with the end behavior of power functions. Another class of functions that grow without bound as x approaches infinity are the log functions: $f(x) = \log_a(x)$. Compare this class of functions with power functions and with exponential functions? Which type of functions "wins" in the limit? Is this always true? Change some of the parameters like you did before and see what happens. Write and justify a theorem (or two) that summarizes your findings.

Also, consider changes in the parameters within these classes of functions. Of course, if we increase the power just a tiny bit, that makes the function bigger, but how much bigger? Enough that $x^4 \underset{\to}{\leq} x^{4.000001}$? How do these changes work in each class of functions? How much larger must p be than q so that $x^q \underset{\to}{\leq} x^p$? What about exponentials and logs?

List an ordering of sorts of log functions, power functions, and exponential functions with a variety of powers/bases that demonstrates all the work you've done.

Note: I just made up the symbol $\underset{\to}{\leq}$, you can use a different symbol just be sure to define it clearly.